Mathematics & Statistics 2021 Research competition

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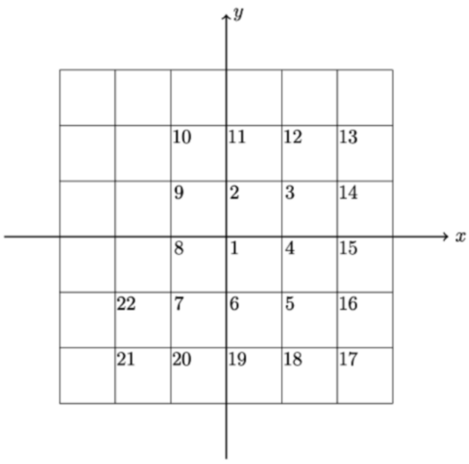
Contents

Question 5: Spiral Cartesian Plane

The problem:

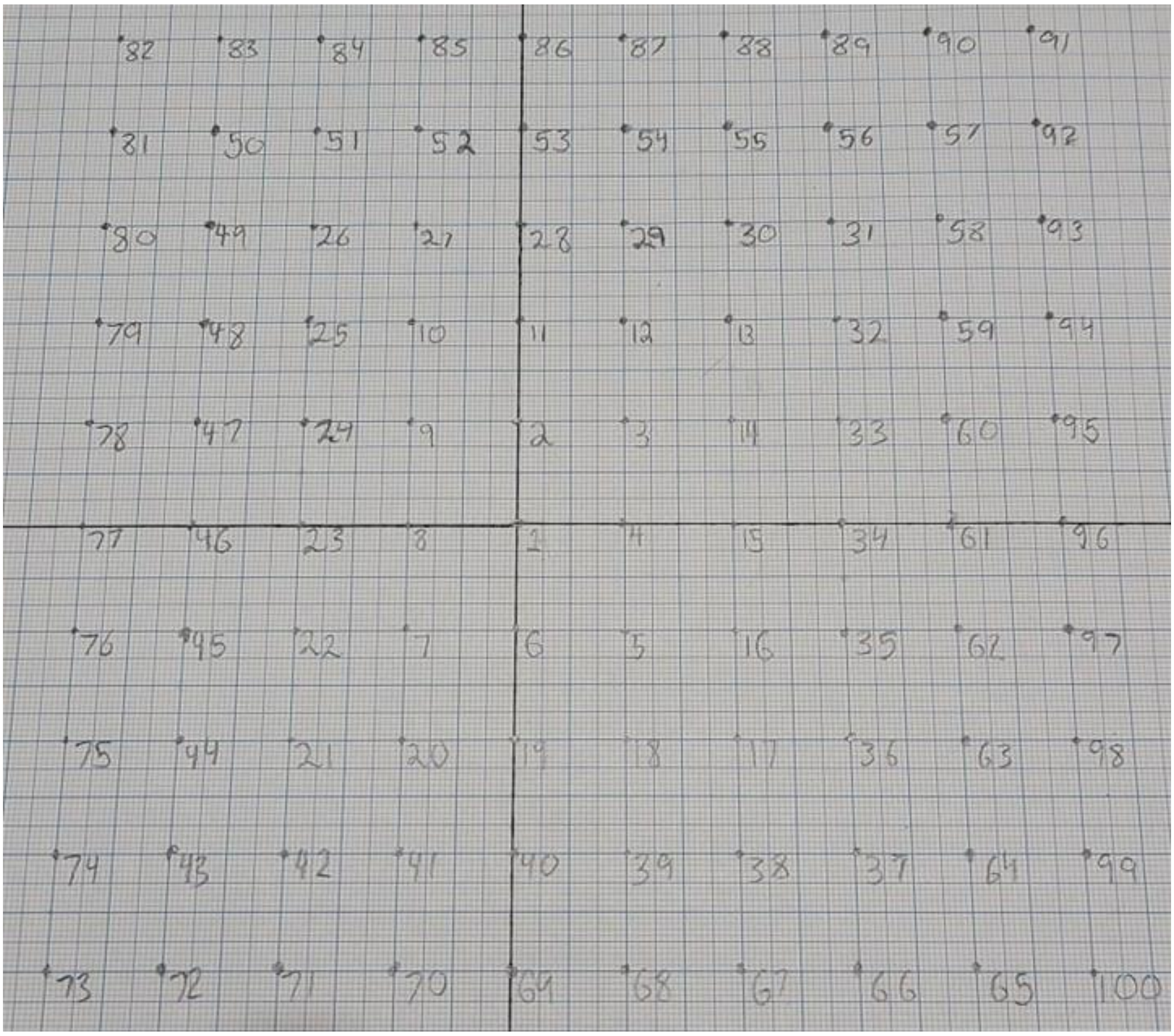
The number 1 is placed at the origin of a cartesian plane. Going in a clockwise spiral, the next integer is added to the grid as shown (for integers 1-22): i.e., number 1 is at (0,0), number 2 is at (0,1) and number 3 is at (1,1).

Construct a method (if one exists) to determine where any integer will lie on the grid.



The Solution:

So first off, to get a better scope of the question and a better idea of how the spiral continues I drew a larger spiral containing the numbers 1 to 100.



After doing so we can see a nice 10x10 square. After a quick inspection there was no obvious pattern or rule that could be found. However, some interesting patterns could be seen in some of the diagonals. Even square numbers could be found on the diagonal from (1,0) to (5,4) and odd square numbers could be seen on the diagonal from (0,0) t0 (-4,4). This rule could be generalised to all square numbers, that they would be find on these diagonals. For example: 144, which is the next even square number would be found on point (5,6) and 121, which is the next odd square number would be found on (-5,5). I continued the spiral in my head to see that this was in fact the case.

However, finding such patterns did not bring me any closer to finding the rule for finding all other numbers so I would have to take a different approach.

Patterns in the x and y axis:

After my attempt at trying to find an obvious pattern was futile, I thought to look for patterns in the positive and negative x and y axis.

Positive x axis:

I first started with the positive x axis. The numbers I had were as follows:

1 4 15 34 61 96

There were no apparent patterns between them, so I looked for the number that I was adding to get to the next number:

1 → 4 → 15 → 34 → 61 → 96

+3 +11 +19 +27 +35

It still felt like there was no pattern, however upon closer inspection I saw that the numbers that I was adding was going up by 8. So, to figure out the next number on the x axis I would have to do 96+43 =139 (because 35+8=43). And to my surprise when I went to go confirm this on the spiral my pattern worked, the next number on the X axis was 139.

Negative x axis:

I did a similar thing with the negative x axis to the positive x axis. The numbers I had were:

1 8 23 46 77

Again, I looked at the number that was added to get the next number

1 → 8 → 23 → 46 → 77

+7 +15 +23 +31

Again, we see the same pattern that we are adding 8 more to the number being added each time. So, the next number that we would add would be +39 giving us 116 (77+39). Once again, I checked this manually to confirm if this was the correct answer.

Positive y axis:

Doing a similar thing with the numbers in the positive y axis:

1 → 2 → 11 → 28 → 53 → 86

+1 +9 +17 +25 +33

As we can see a very similar pattern of adding 8 to the number that is being added each time. Thus, to find the next number in this pattern we would add 41 giving us 127 (86+41=127). Again, this answer was confirmed by the manual process of filling in the numbers of the spiral.

Negative y axis:

Doing the same thing as above but with the negative y axis:

1 → 6 → 19 → 40 → 69

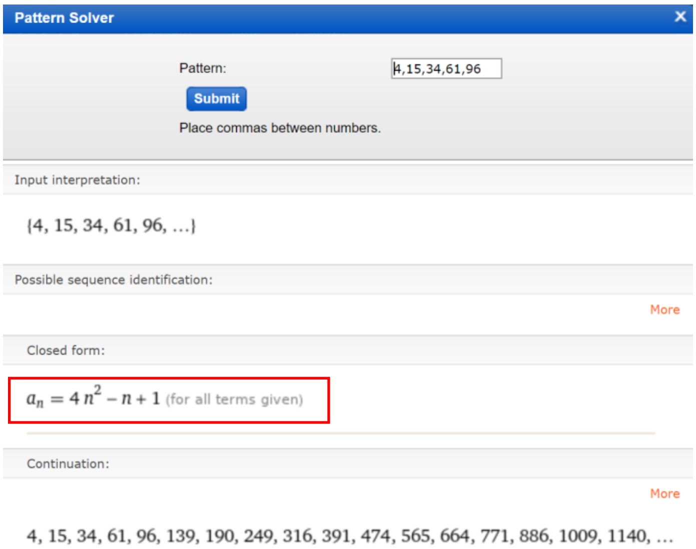
+5 +13 +21 +29

The same pattern applies for this axis also where we are adding 8 to the number being added each time. Thus, for next number we would add 37 giving us 106 (69+37). This was confirmed by referring to the manual spiral that was created.

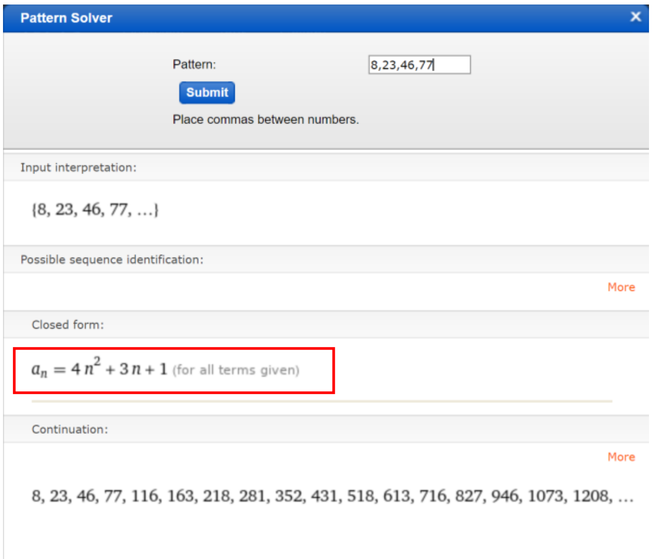
Summary of axis patterns:

So now that I was able to find patterns for each axis, I needed to find a rule for any number on the axis. For this I used Wolfram Alpha's pattern solver which gave me a rule for each axis. (Note that I did not include the number 1 which was on the origin. Where a_n is the value of the spiral and 'n' is the x or y axis value)

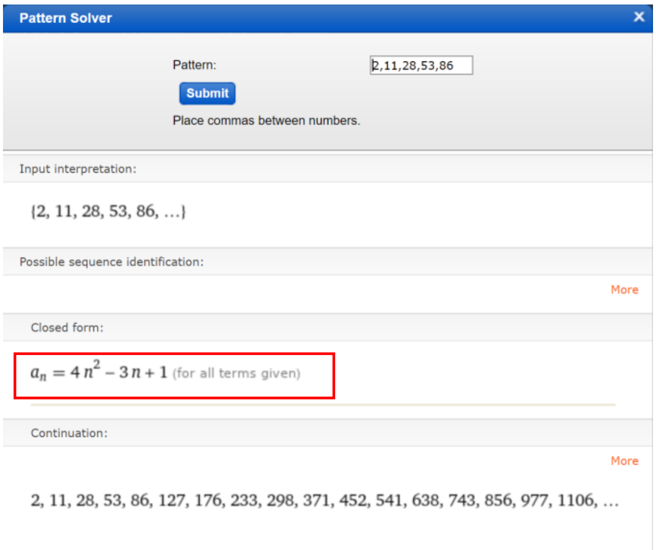
Positive x axis rule:



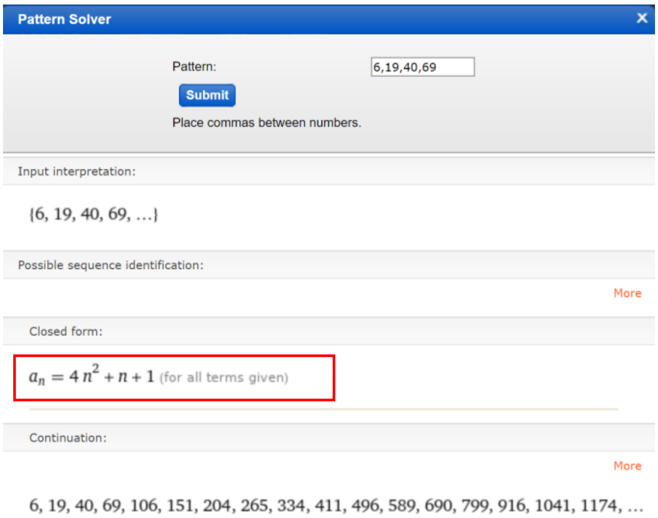
Negative x axis rule:



Positive y axis rule:



Negative y axis rule:



All rules for each axis:

(Let V be the value of the spiral)

Positive x axis: $V = 4x^2 - x + 1$

Negative x axis: $V = 4(-x)^2 + 3(-x) + 1 = 4x^2 - 3x + 1$

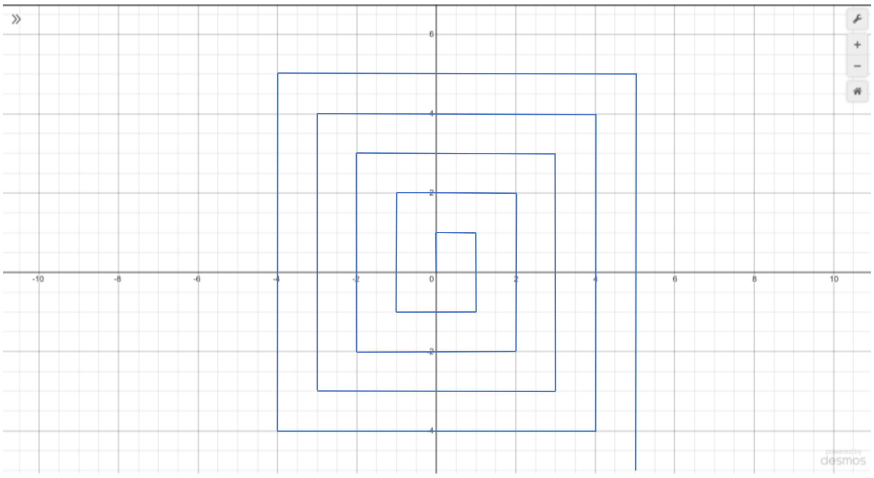
Positive y axis: $V = 4y^2 - 3y + 1$

Negative y axis: $V = 4(-y)^2 + (-y) + 1 = 4y^2 - y + 1$

The next step: Finding the general rule

So now that I was able to establish rules for the numbers on the axis, I needed to figure out a rule for any number on the spiral. This was a much harder task, after countless hours drawing spirals and looking for patterns and rules, I finally was able to figure out a relationship between the numbers in each quadrant and the axis.

To do so we need to first understand the relationship between the numbers of the spiral and the axes.



After drawing the spiral what I noticed is that the spiral will always go through the axis, never with it (except for right at the beginning).

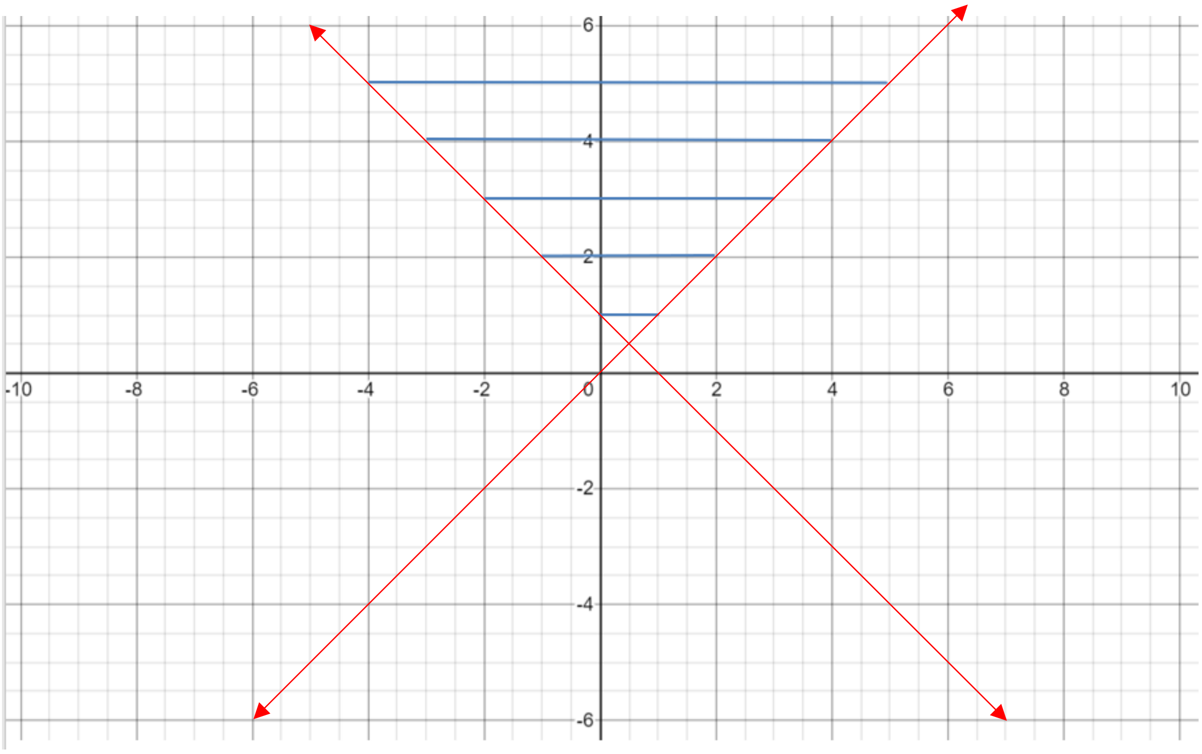
That would mean that that consecutive numbers of the spiral (which contain a number which lies on the one of the axes) would be next to each other. For example the number 11 is at (0,2)- on the y axis. Number 10 is before to its left and 11 and 12 after it to its right.

Similarly, 13, 14, 15 ,16 ,17 all lie on a line where 15 goes through the x axis.

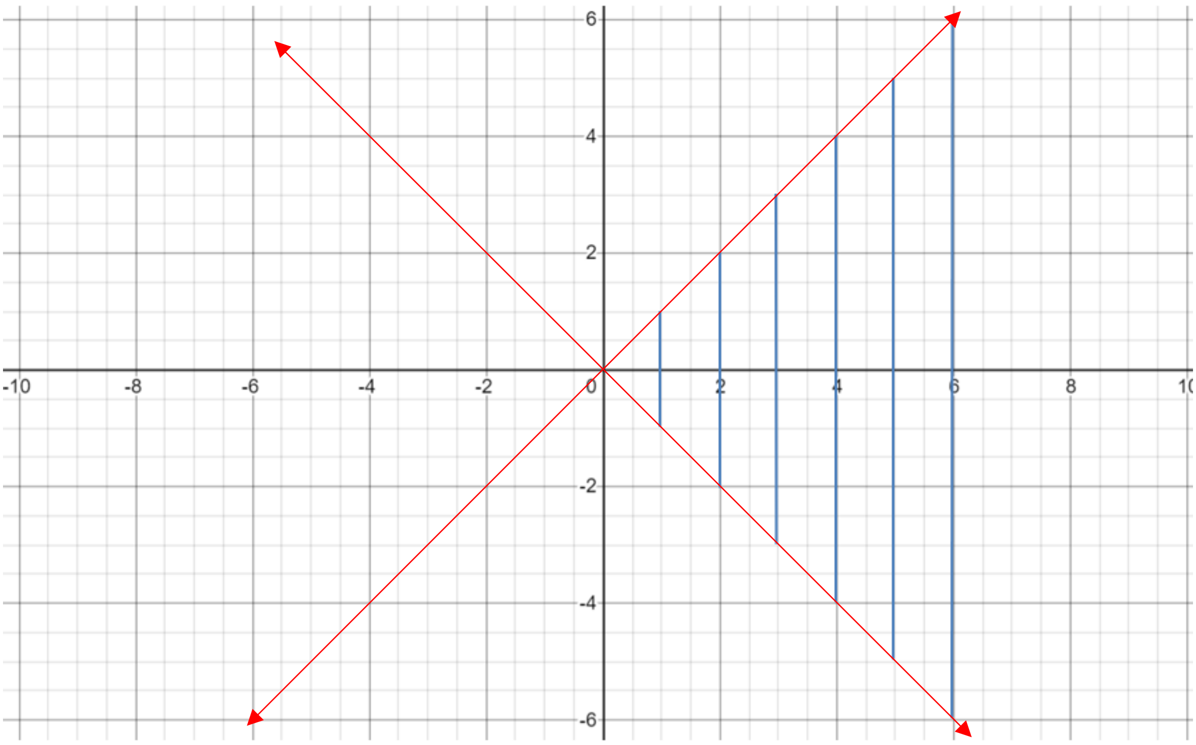
Thus, if we know the values of the numbers that lie on the axes, we will be able to determine the numbers that lie on the line of the spiral that goes through the axes.

The lines of the spiral are all shown in the following pages separated by which part of the axes that it goes through.

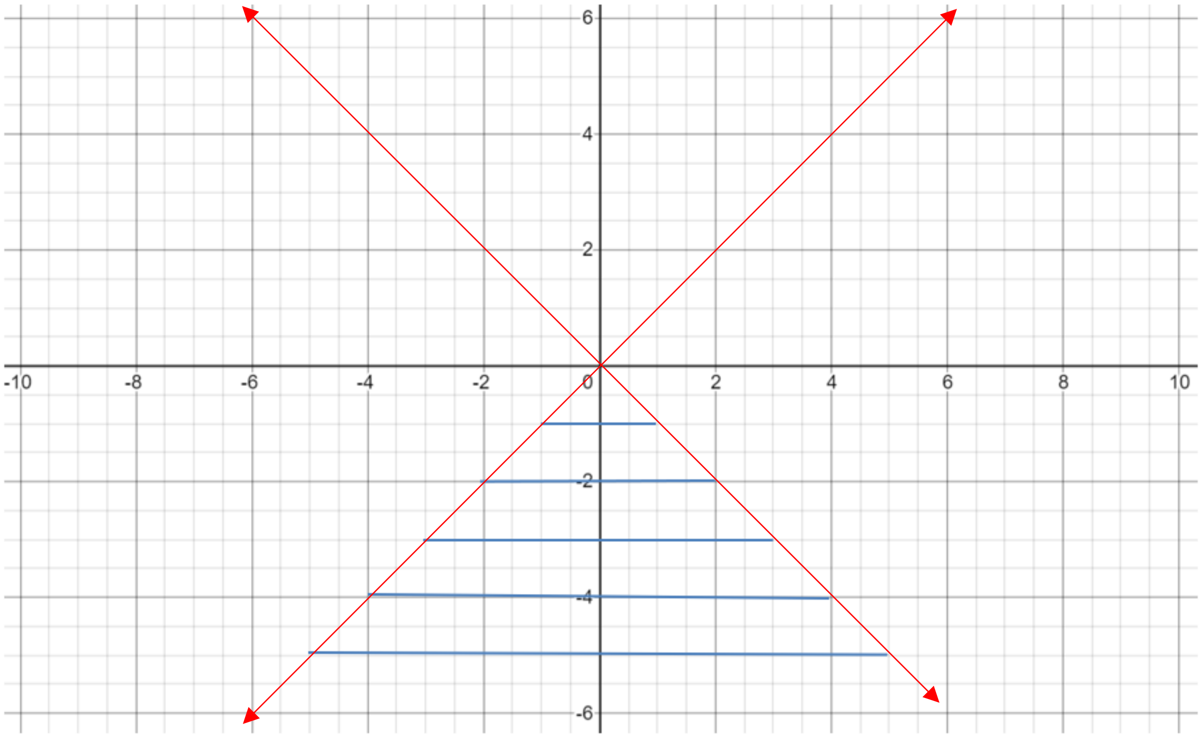
Positive y-axis:



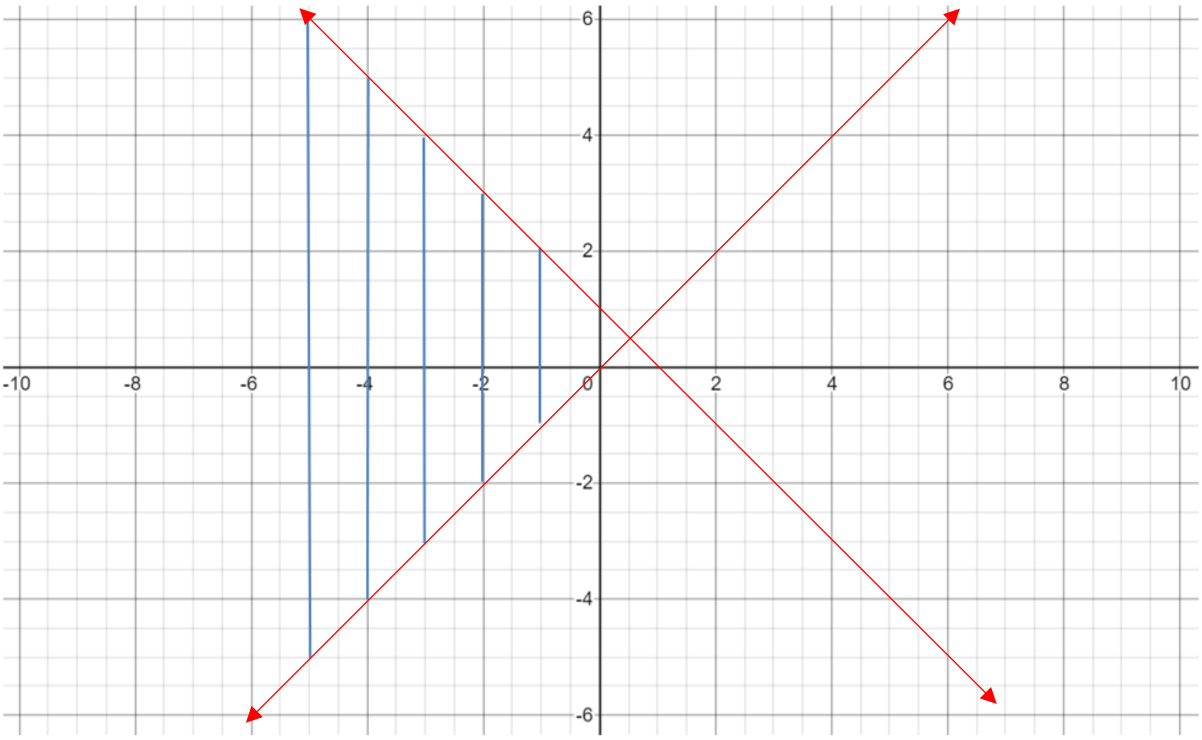
Positive x-axis:



Negative y-axis:



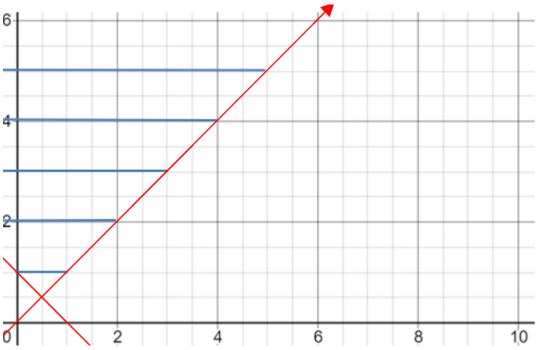
Negative x-axis:

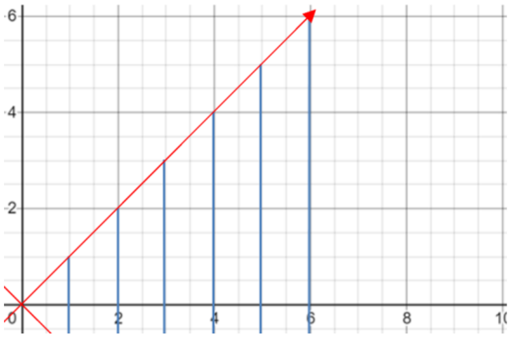


Rules in Quadrants:

As we can see from the graphs above, each quadrant has both vertical and horizontal lines. The horizontal lines will have numbers consecutive to the y axis value (of the spiral) it passes through, and the vertical lines will have numbers consecutive to the x axis value it passes through. This means the only difference between the values anywhere on the spiral and the value on the axes is how far from the axis that the line passes through is it.

1st Quadrant:





From before:

The value of the spiral on the positive y axis = $4y^2 -3y +1$
The value of the spiral on the positive x axis = $4x^2 -x +1$

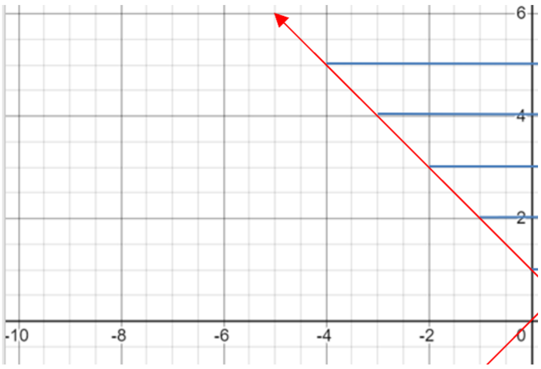
So, any value to the right of the positive y axis is 1 more than the axis value. And it continues going up by one until we reach the corner of the spiral which is on the line y=x

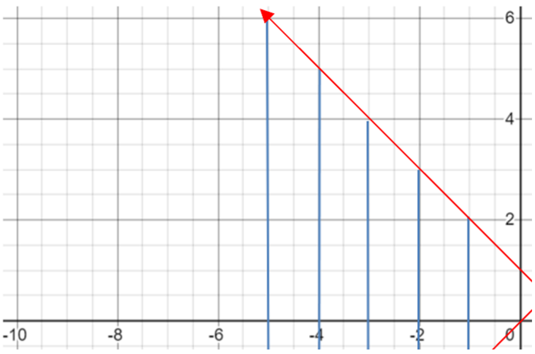
Thus, the formula is **V=** $\mathbf{4y^2 -3y +1 +x}$ (for $y \geq x$, $x > 0$ and $y > 0$)

Similarly, any value above the positive x axis is one less than the x axis value and it keeps decreasing by 1 until we reach the corner of the spiral.

Thus, the formula is **V=** $\mathbf{4x^2 -x +1 -y}$ (for $y < x$, $x > 0$ and $y > 0$)

2nd Quadrant:





From before:

The value of the spiral on the positive y axis = $4y^2 -3y +1$
The value of the spiral on the negative x axis = $4x^2 -3x +1$

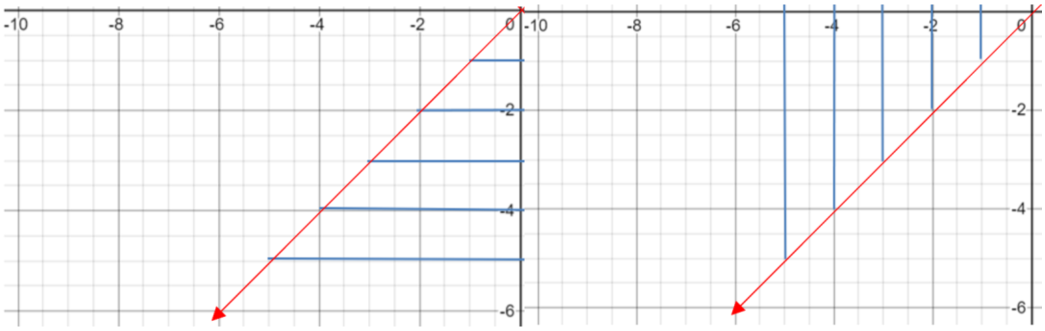
Thus, any value to the left of the positive y axis is 1 less than the axis value. And it continues decreasing by 1 until we reach the corner on the line $y= -x +1$

Thus, the formula is **$V= 4y^2 -3y +1 +x$** (for $y \geq -x+1$, $x<0$ and $y> 0$)

Similarly, when we are above the negative x axis the value is 1 more than that of the axis. The values keep increasing by 1 until we reach the corner of the spiral.

Thus, the formula is **$V= = 4x^2 -3x +1 +y$** (for $y < -x+1$, $x<0$ and $y> 0$)

3^{rd} Quadrant:



From before:

The value of the spiral on the negative y axis = $4y^2 +y +1$
The value of the spiral on the negative x axis = $4x^2 +3x +1$

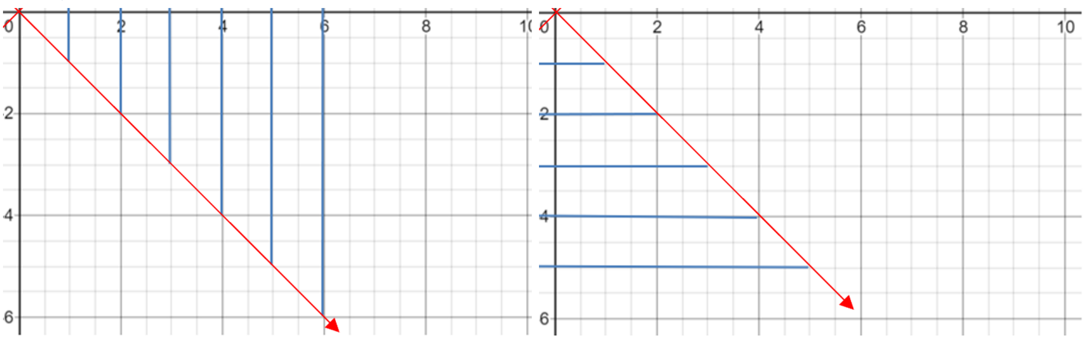
Any value to the left of the negative x axis is one more than that of the axis value. And it continues to increase by 1 until we reach the corner of the spiral on the line $y=x$

Thus, the formula is **$V= 4y^2 -y +1 -x$** (for $y<-x$, $x<0$ and $y<0$)

Similarly, when we are below the negative x axis the value is 1 less than that of the axis. The values keep decreasing by one until we reach the corner of the spiral.

Thus, the formula is **$V=4x^2 -3x +1 +y$** (for $y \geq -x$, $x<0$ and $y<0$)

4^{th} Quadrant:



From before:

The value of the spiral on the positive x axis = $4x^2 - x + 1$
The value of the spiral on the negative y axis = $4y^2 + y + 1$

Any value below the positive x axis is 1 more than the axis value. This continues to increase until we reach the corner of the spiral which lies on the line y=-x.

Thus, the formula is **$V = 4x^2 - x + 1 - y$** (For $y \geq -x$, $x>0$, $y<0$)

Similarly, any value to the right of the negative y axis is consecutive to the y axis value and decreases by one until we reach the corner.

Thus, the formula is **$V = 4y^2 - y + 1 - x$** (For $y < -x$, $x>0$, $y<0$)

Summary/ final rules:

Thus, the formula is **$V = 4y^2 - 3y + 1 + x$** (for $y \geq x$, $x > 0$ and $y > 0$)
Thus, the formula is **$V = 4x^2 - x + 1 - y$** (for $y < x$, $x > 0$ and $y > 0$)

Thus, the formula is **$V = 4y^2 - 3y + 1 + x$** (for $y \geq -x+1$, $x<0$ and $y>0$)
Thus, the formula is **V= = $4x^2 - 3x + 1 + y$** (for $y < -x+1$, $x<0$ and $y>0$)

Thus, the formula is **$V = 4y^2 - y + 1 - x$** (for $y<-x$, $x<0$ and $y<0$)
Thus, the formula is **$V=4x^2 - 3x + 1 + y$** (for $y \geq -x$, $x<0$ and $y<0$)

Thus, the formula is **$V = 4x^2 - x + 1 - y$** (For $y \geq -x$, $x>0$, $y<0$)
Thus, the formula is **$V = 4y^2 - y + 1 - x$** (For $y < -x$, $x>0$, $y<0$)