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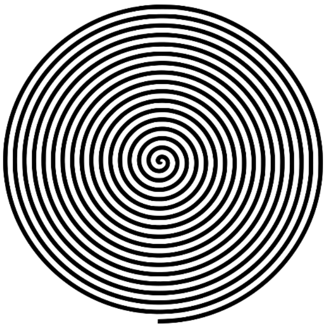
The Cartesian Spiral



The Cartesian Spiral is a number plane that shows consecutive positive integers that are added in a clockwise spiral with the first number 1 at the origin (0, 0). The number 2 is at (0, 1) and number 3 is at (1, 1). See the diagram below:

			y				
		10	11	12	13		
		9	2	3	14		x
		8	1	4	15		
	22	7	6	5	16		
	21	20	19	18	17		

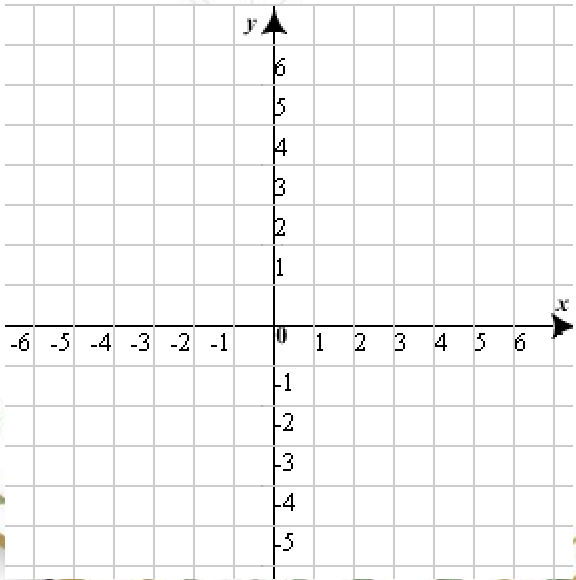


Diagram 1

Below, the positive integers are continued up to number 169. While some random patterns exist within the spiral, the one that is most fascinating is the position of the square numbers which forms the main basis of exploring this clockwise spiral. The odd square numbers are highlighted in grey while the even square numbers are highlighted in blue.

y

		169	122	123	124	125	126	127	128	129	130	131	132	133		
		168	121	82	83	84	85	86	87	88	89	90	91	134		
		167	120	81	50	51	52	53	54	55	56	57	92	135		
		166	119	80	49	26	27	28	29	30	31	58	93	136		
		165	118	79	48	25	10	11	12	13	32	59	94	137		
		164	117	78	47	24	9	2	3	14	33	60	95	138		
		163	116	77	46	23	8	1	4	15	34	61	96	139		
		162	115	76	45	22	7	6	5	16	35	62	97	140		
		161	114	75	44	21	20	19	18	17	36	63	98	141		
		160	113	74	43	42	41	40	39	38	37	64	99	142		
		159	112	73	72	71	70	69	68	67	66	65	100	143		
		158	111	110	109	108	107	106	105	104	103	102	101	144		
		157	156	155	154	153	152	151	150	149	148	147	146	145		

x

$5^2=25$ $9^2=81$

Odd Square Numbers

If we observe the Cartesian Spiral, notice that the odd square numbers are located in the second quadrant in a perfectly diagonal pattern moving upwards. The odd square numbers are ascending in the North-West direction starting at number 1.

Table 1

Odd Square Number	Value	Position on Cartesian Plane
1^2	1	(0, 0)
3^2	9	(-1, 1)
5^2	25	(-2, 2)
7^2	49	(-3, 3)
9^2	81	(-4, 4)
n^2		$(\frac{1-n}{2}, \frac{n-1}{2})$

$7^2 = 49$ $11^2 = 121$

Looking at the values in the above table, a rule can be obtained for the location of any odd square number. For example, we can easily locate 625, an odd square number where n = 25 since 252 = 625. Substituting n = 25 in the rule gives us (-12, 12).

$12^2 = 144$

$4^2 = 16$

Even Square Numbers

If we observe the Cartesian Spiral, notice that the even square numbers are located in the fourth quadrant in a perfectly diagonal pattern, moving downwards. The odd square numbers are ascending in the South-East direction along the diagonal starting at the top left at number (4 the first even square number).

TABLE 2

Even Square Number	Value	Position on Cartesian Plane
2^2	4	(1, 0)
4^2	16	(2, -1)
6^2	36	(3, -2)
8^2	64	(4, -3)
10^2	100	(5, -4)
n^2		$(\frac{n}{2}, 1 - \frac{n}{2})$

$10^2 = 100$

Looking at the values in the above table, a rule can be obtained for the location of any even square number. For example, we can easily locate 324, an even square number where n = 18 since 182 = 324. Substituting n = 18 in the rule gives us (9, -8).

$2^2 = 4$

Is there a method where we can confidently and accurately predict where exactly any positive integer will lie on the Cartesian spiral?

Before we answer the question above, let us find out:

How does the Spiral continue once we locate a square number?

First let's consider the continuing pattern of odd square numbers.
Let's lock in number 9, an Odd Square Number (n = 3 since 32 = 9)

Table 3

Value of n (Odd)	Square number obtained (n2)	Pattern on the grid to reach the next odd square number
3	9	1 UP, 3 RIGHT, 4 DOWN, 4 LEFT, 4 UP
5	25	1 UP, 5 RIGHT, 6 DOWN, 6 LEFT, 6 UP
7	49	1 UP, 7 RIGHT, 8 DOWN, 8 LEFT, 8 UP
n	n^2	1 UP, n RIGHT, (n+1) DOWN, (n+1) LEFT, (n+1) UP

Now let us consider the continuing pattern of even square numbers.

Let's lock in number 4, an even Square Number (n = 2 since 22 = 4)

Table 4

Value of n (Even)	Square number obtained (n2)	Pattern on the grid to reach the next even square number
2	4	1 DOWN, 2 LEFT, 3 UP, 3 RIGHT, 3 DOWN
4	16	1 DOWN, 4 LEFT, 5 UP, 5 RIGHT, 5 DOWN
6	36	1 DOWN, 6 LEFT, 7 UP, 7 RIGHT, 7 DOWN
n	n^2	1 DOWN, n LEFT, (n+1) UP, (n+1) RIGHT, (n+1) DOWN

What other patterns are obvious related to the position of a square number?

Table 5

Value of n (Odd)	Square number obtained (n2)	Pattern
3	9	Going down up to maximum two units, the square number keeps decreasing by 1 (to give 8 and 7)
5	25	Going down up to maximum four units, the square number keeps decreasing by 1 (24, 23, 22 and 21)
7	49	Going down up to maximum six units, the square number keeps decreasing by 1 (48, 47, 46, 45, 44, 43)
n	n^2	**Go down up to maximum (n - 1) units, the square number keeps decreasing by 1 (up to n - 1 entries)**

See **Diagram 2**

Table 6

Value of n (Even)	Square number obtained (n2)	Pattern
2	4	One unit directly below: Square number increases by 1 (4+1=5) Up to one unit maximum vertically above: square number decreases by 1 (4 - 1= 3)
4	16	One unit directly below: Square number increases by 1 (16+1=17) Up to three units maximum directly above: square number decreases by 1 to give (15, 14, 13)
6	36	One unit directly below: Square number increases by 1 (36+1=37) Up to five units maximum directly above: square number decreases by 1 to give (35, 34, 33, 32, 31)
n	n2	One unit directly below: square number increases by n + 1 Up to (n - 1) units' maximum directly above: square number decreases by 1 to give (15, 14, 13) decrease by 1 (up to n - 1 entries)

See **Diagram 2**

Diagram 2

							y									
		169	122	123	124	125	126	127	128	129	130	131	132	133		
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		167	120	81	50	51	52	53	54	55	56	57	92	135		
		166	119	80	49	26	27	28	29	30	31	58	93	136		
		165	118	79	48	25	10	11	12	13	32	59	94	137		
		164	117	78	47	24	9	2	3	14	33	60	95	138		x
		163	116	77	46	23	8	1	4	15	34	61	96	139		
		162	115	76	45	22	7	6	5	16	35	62	97	140		
		161	114	75	44	21	20	19	18	17	36	63	98	141		
		160	113	74	43	42	41	40	39	38	37	64	99	142		
		159	112	73	72	71	70	69	68	67	66	65	100	143		
		158	111	110	109	108	107	106	105	104	103	102	101	144		
		157	156	155	154	153	152	151	150	149	148	147	146	145		

Let Us Test It Out

Where would the number 384 be found on the Cartesian number plane?

182 = 324 (n = 18)

192 = 361 (n = 19)

384 is closer to 361. We can use the rule in Table 1 to see where 361 is located. Using the rule $(\frac{1-n}{2}, \frac{n-1}{2})$ where n = 19 we get (-9, 9)

From 361 to 384 we have to travel 23 units. Then referring to Table 3, the travel pattern for n = 19 is:

1 UP

Coordinates will change to (-9, 10) [Here we find the number 362]

19 RIGHT

Coordinates will change to (10, 10) [Here we find the number 372]

20 DOWN {but we only have to move 3 DOWN as we have already travelled 20 units}

So, 3 DOWNS will get us to (10, 7).

Hence the number 384 is located at (10, 7)

Another Example:

Where would the number 309 be found on the Cartesian number plane?

Let us use 182 = 324 (n = 18)

We can use the rule in Table 2 to see where 324 is located. Using the rule ((n)/2,1- (n)/2) where n = 18 we get (9, -8). Use the pattern movement from an even square number as described in Table 6, the numbers descend in a directly upward direction up to 17 units.

We only need 15 movements to go from 324 to 309. The coordinates will then change to (9, -8+15) = (9, 7)

Hence the number 309 is located at (9, 7)

Conclusion:

The Cartesian Clockwise Spiral is a highly interesting array of numbers that lets us predict where numbers will be located using the number pattern of where the square numbers are located.