THE UNIVERSITY OF MELBOURNE

Mathematics and Statistics Research Competition 2021

Wild Card Project: To Infinity and Beyond…

BY: TAWQEER SAIYED (YR12)

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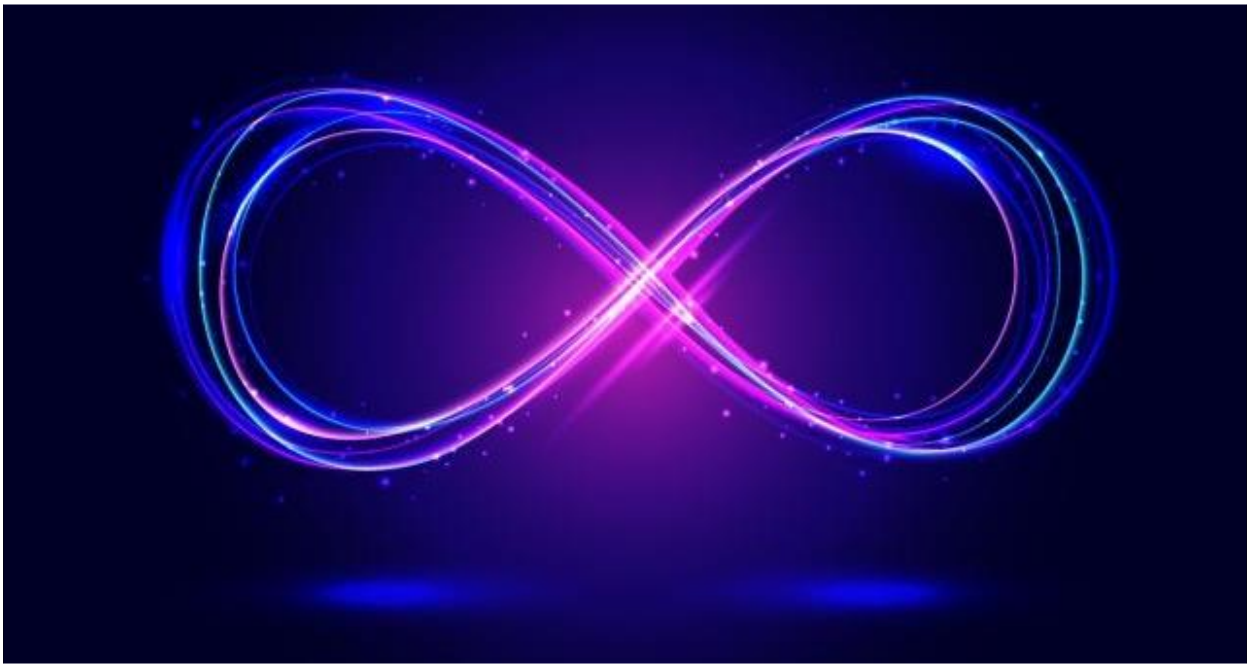


INTRODUCTION

This is a report on the 2021 mathematics and statistics research competition of the University of Melbourne. The senior question I have chosen to do is Q8: The Wild Card Project. The stimulus is given to be "To Infinity and Beyond." The topics which came to mind instantly when I first read this stimulus were two things; **exponential functions and calculus involving integration**.

I thought of exponential functions at first as these functions commonly are seen to rapidly rise towards **infinity**. The most common exponential graphs embedded in most people's heads feature a very steep curve, which rises indefinitely all the way towards infinity. This is a prime idea relating to infinity. Another thought which entered my mind was the calculus of integration. This is because of the fundamental theorem of calculus, which implies that the integral of a function is the area under the curve itself, and this area is calculated by finding the sum of **infinitesimally** small increments of the function's value. This key concept of integration being the summation of an **infinite** amount of values, makes it another perfect candidate for the stimulus involving infinity.

I was unable to decide which one of these two to pick for my research question. Until I had a 'eureka' moment and decided to do both! Finding the **infinite sum** of a function which exemplifies **infinity** will cover the "and beyond" part of the question. Hence, this report will be centred around the integral calculus of exponential functions. The research questions were made soon after, following my curiosity.

Self-Made Research Questions:

1.) What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary power x?

$$\int a^x dx\,?$$

2.) What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary **linear function** of x?

$$\int a^{mx+b} dx\,?$$

3.) What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary **function** of x, where the **derivative** of the function is the multiplier of the constant?

$$\int f'(x)a^{f(x)} dx\,?$$

(The integral of $a^{f(x)}$ by itself is not possible when $f(x)$ is higher than order 1, hence will not be covered in this report, only the case when the derivative of the function is next to the constant)

This report aims to:

- Introduce exponential functions, and the process of integration, along with relevant formulas and laws required for the report.
- Provide complete, comprehensive, and clear answers to the above research questions, including full derivations and numerical examples where needed.
- Include the use of technology as screenshots (using Casio ClassPAD fx-CP400 graphic calculator)
- Include full in-text referencing, along with professional mathematical format for working out.
- Include a comprehensive summary and conclusion of results found.

EXPONENTIAL FUNCTIONS

The most basic exponential functions are given in the general form $y = f(x) = a^x$, with a being a positive constant and x being a part of the set of real numbers.

Arguably the most common exponential function is when $a = e \approx 2.71$, as here the derivative of the resultant function is the function itself. This property is the reason why this function is readily used in calculus and to model many real-life applications. It will be needed in our report. Refer to FIGURE 1.[1]

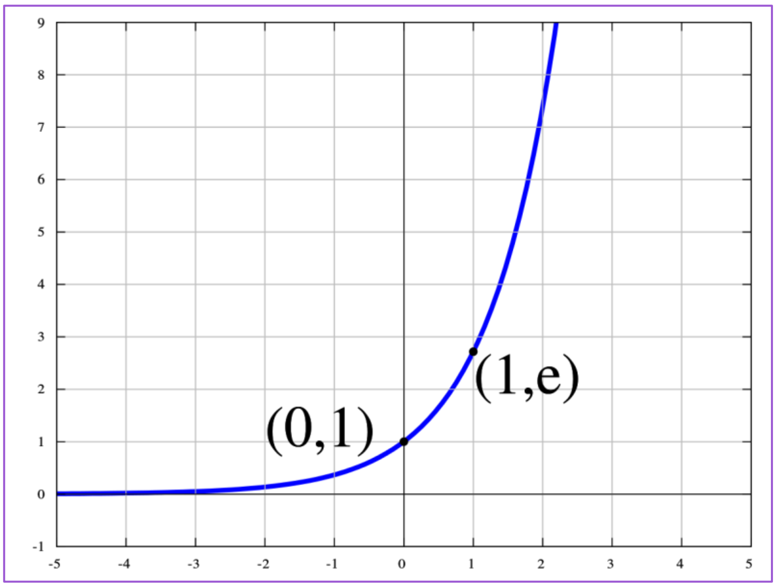


FIGURE 1: a simple graph showing the function $y = e^x$, highlighting the extremely steep gradient of the exponential function as it rises towards infinity.

INTEGRAL CALCULUS

One of the implications of the fundamental theorem of calculus, which includes the statement that integration and derivation are inverse processes of each other, is that integration represents the area under a curve. This was derived by finding that when considering the sum of infinitesimal heights of a curve above the horizontal axis, the result was the definite integral of the function itself. Hence, the discovery of integral calculus made it much easier to calculate the area under curves of functions, through the use of infinite sums. Refer to Figure 2[2] for a visual representation.

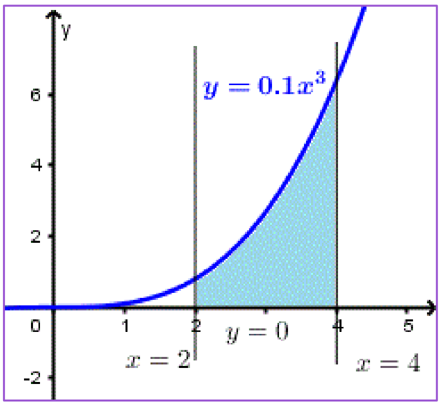


FIGURE 2: A representation of the area under a curve of an arbitrary exponential function. We can see that the area is calculated using the sum of the infinitesimally small increments of heights of the function in the given bounds.

[1] Adapted from https://en.wikipedia.org/wiki/Exponential_function
[2] https://www.analyzemath.com/calculus/Integrals/area_under_curve.html

FORMULAS

This section hopes to introduce all the formulas that will be very useful for the report.

LOGARITHMIC LAWS

Extracted from the WA Mathematics Methods Year 12 Formula Sheet[3], all the logarithmic laws are presented here. These laws will prove very handy later in the report.

Logarithms

$x = \log_a b \Leftrightarrow a^x = b$	$a^{\log_a b} = b$ and $\log_a(a^b) = b$
$\log_a mn = \log_a m + \log_a n$	$\log_a \frac{m}{n} = \log_a m - \log_a n$
$\log_a(m^k) = k \log_a m$	$\log_e x = \ln x$

INTEGRATION

Provided below are the rules of the integration of exponential functions with the base e:

$$\int e^x \, dx = e^x + c$$

The integral of e^x is itself with an added constant.

$$\int e^{ax+c} \, dx = \frac{e^{ax+c}}{a} + c$$

The integral of e to a linear power is itself divided by the derivative of the power with an added constant.

$$\int f'(x)e^{f(x)} \, dx = e^{f(x)} + c$$

The integral of e to the power of a function multiplied by the derivative of that function is itself with an added constant.

QUESTION 1

What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary power x? ($\int a^x dx$)

This question was created out of curiosity as I had recently learned about exponential and logarithmic functions in class, however I always wondered what if we had the integral of a constant other than e, which is all we have worked with? This section will deal with just that, by deriving the general formula for this form of an integral.

[3] https://senior-secondary.scsa.wa.edu.au/

Since we know how to deal with exponential functions in the form $e^{f(x)}$, let us convert the above constant involving e, using logarithms. Logarithmic laws tell us that $y = x^{log_x(y)}$

$$\therefore a = e^{log_e(a)}$$

Writing $log_e(x)$ as $\ln(x)$ (natural logarithm):

$$a = e^{ln(a)}$$

$$\text{Thus } a^x = \left(e^{ln(a)}\right)^x$$

$$\therefore a^x = e^{xln(a)} \ (as\ (k^m)^n = k^{mn})$$

Hence the integral becomes:

$$\int e^{xln(a)}\, dx$$

Noticing that $\ln(a)$ is a constant, we essentially have the form e^{kx} where k is a constant. Hence we may apply the integration rule of $\int e^{px+q}\, dx = \frac{e^{px+q}}{p} + c$:

$$\therefore \int e^{xln(a)}\, dx = \frac{e^{xln(a)}}{\ln(a)} + c$$

Using $xlog(b) = \log(b^x)$:

$$= \frac{e^{ln(a^x)}}{\ln(a)} + c$$

$$= \frac{a^x}{\ln(a)} + c$$

$$\therefore \int a^x dx = \frac{a^x}{\ln(a)} + c$$

Hence, the indefinite integral of a^x with respect to x is a^x divided by the natural log of a

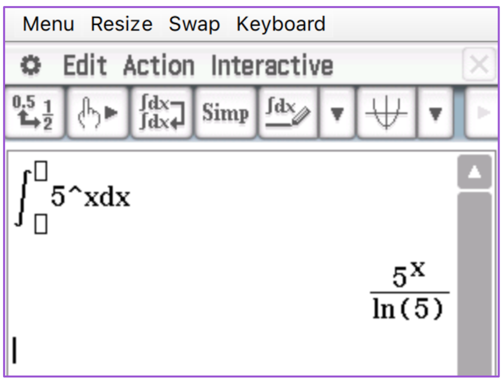
NUMERICAL EXAMPLE

In order to test the formula, let us consider a numerical example. Consider $\int 5^x\, dx$.

Using the formula above we get:

$$\int 5^x dx = \frac{5^x}{\ln(5)} + c$$

Now testing this on a calculator:



Screenshot from ClassPAD Calculator, giving the same answer as the formula (without the constant). This strengthens the validity of our conjecture.

QUESTION 2

What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary **linear function** of x? ($\int a^{mx+b} dx$)

This question seeks to go a step ahead from the first, to see if we can extend our findings.

As before, let us begin by converting the constant to a power of e, so we can work with it. From the first question:

$$a = e^{ln(a)}$$

$$\therefore a^{mx+b} = \left(e^{ln(a)}\right)^{mx+b}$$

$$= e^{\ln(a)(mx+b)}$$

$$= e^{mln(a)x+ln(a)b}$$

$$\therefore \int a^{mx+b}\, dx = \int e^{mln(a)x+ln(a)b} dx$$

However, since all $\ln(a)$, m and n are constants, we see that $mln(a)$ and $\ln(a)b$ must also be constants. Thus we can rewrite this in a simplified form by substituting in new variables representing these constants:

$Let\ k = mln(a)$ and $n = \ln(a)\, b,$ where k and n are real constants.

$$\therefore e^{mln(a)x+ln(a)b} = e^{kx+n}$$

We can now apply the integration rule $\int e^{ax+c}\, dx = \frac{e^{ax+c}}{a} + c$:

$$\int e^{kx+n}\, dx = \frac{e^{kx+n}}{k} + c$$

Substituting back $k = mln(a)$ and $n = \ln(a)\, b$:

$$= \frac{e^{mln(a)x+\ln(a)b}}{mln(a)} + c$$

Using the law $xlog(b) = \log(b^x)$:

$$= \frac{e^{\ln(a^{mx})+\ln(a^{b})}}{\ln(a^{m})} + c$$

Using the law $\log_a(x) + \log_a(y) = \log_a(xy)$:

$$= \frac{e^{\ln(a^{mx}\times a^{b})}}{\ln(a^{m})} + c$$

Using the property $q^x \times q^y = q^{x+y}$:

$$= \frac{e^{\ln(a^{mx+b})}}{\ln(a^{m})} + c$$

Simplifying using $y = x^{log_x(y)}$:

$$= \frac{a^{mx+b}}{\ln(a^{m})} + c$$

$$\therefore \int a^{mx+b}\, dx = \frac{a^{mx+b}}{\ln\left(a^{m}\right)} + c$$

Hence, indefinite integral of a positive constant a raised to an arbitrary **linear function** of x, is itself divided by the natural log of a to the power of the derivative of the linear power.

NUMERICAL EXAMPLE

In order to test the formula, let us consider a numerical example. Consider $\int 2^{3x+4}\, dx$.

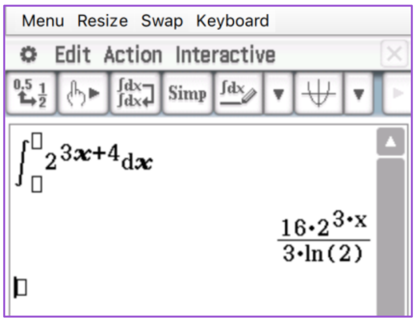
Using the formula above we get:

$$\int 2^{3x+4}\, dx = \frac{2^{3x+4}}{\ln(2^3)} + c$$

$$= \frac{2^{3x} \times 2^4}{3\ln(2)} + c$$

$$= \frac{16 \times 2^{3x}}{3\ln(2)}$$

Now testing this on a calculator:



Screenshot from ClassPAD Calculator, giving the same answer as the formula (without the constant). This strengthens the validity of our conjecture.

QUESTION 3

What is the general formula for the indefinite integral of a positive constant a raised to an arbitrary **function** of x, where the **derivative** of the function is the multiplier of the constant? ($\int f'(x)a^{f(x)}dx$)

This question again seeks to go a yet another step ahead from the first, to see if we can extend our findings.

As before, let us begin by converting the constant to a power of e, so we can work with it. From the first question:

$$a = e^{\ln(a)}$$

$$\therefore a^{f(x)} = \left(e^{\ln(a)}\right)^{f(x)}$$

$$= e^{f(x)\ln(a)}$$

$$\therefore \int f'(x)\, a^{f(x)} dx = \int f'(x)\, e^{f(x)\ln(a)} dx$$

We now arrive at the crux of the working in this next step. First we realise that $\ln(a)$ is a constant, and hence the power of the e on the right hand side of the above equation is simply a constant multiplied by a function. Since functions are not a set number – they are a collection of **variables** – we cannot just assume that the entire term will be constant, and proceed.

We need to investigate what a constant multiplied by a function would be. Let the constant be called k and the function be $g(x)$.

$$\therefore kg(x)$$

Since we want to find the integral of a function within a power of e, we know we are dealing with the rule $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$. Since we see the derivative of the function being needed to be at the front, let us investigate the derivative of $kg(x)$ to find what we require.

$$\frac{d}{dx}\left(kg(x)\right)$$

$$= k\frac{d}{dx}\left(g(x)\right)$$

$$= kg'(x)$$

Thus we can conclude that the derivative of a constant multiplied by a function is equal to the same constant multiplied by the derivative of the original function. This is useful to us as we now know that for our integral above, we require the constant in front of the e:

i.e. we require $\int \ln(a)\, f'(x)\, e^{f(x)\ln(a)} dx$

But we cannot add extra things to our expression, so to compensate we must divide what we added:

$$\int f'(x)\, e^{f(x)\ln(a)} dx = \frac{1}{\ln(a)} \int \ln(a)\, f'(x)\, e^{f(x)\ln(a)} dx$$

Now we may directly apply the formula

$\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + c$:

$$\frac{1}{\ln(a)} \int \ln(a)\, f'(x)\, e^{f(x)\ln(a)} dx$$

$$= \frac{1}{\ln(a)} \left[e^{f(x)\ln(a)} \right] + c$$

$$= \frac{e^{f(x)\ln(a)}}{\ln(a)} + c$$

Using the law $x\log(b) = \log(b^x)$:

$$= \frac{e^{\ln(a^{f(x)})}}{\ln(a)} + c$$

Simplifying using $y = x^{\log_x(y)}$:

$$= \frac{a^{f(x)}}{\ln(a)} + c$$

$$\therefore \int f'(x) a^{f(x)} dx = \frac{a^{f(x)}}{\ln(a)} + c$$

Hence, the indefinite integral of a positive constant a raised to an arbitrary function of x, where the derivative of the function is the multiplier of the constant, is the constant with the function as a power divided by the natural log of the constant.

NUMERICAL EXAMPLE

In order to test the formula, let us again consider a numerical example. Consider $\int f'(x)3^{f(x)}\,dx$ where $f(x) = 4x^3 - 6x^2 + 5x - 2$

Using the formula above

Using the power rule:

If $f(x) = 4x^3 - 6x^2 + 5x - 2$ then

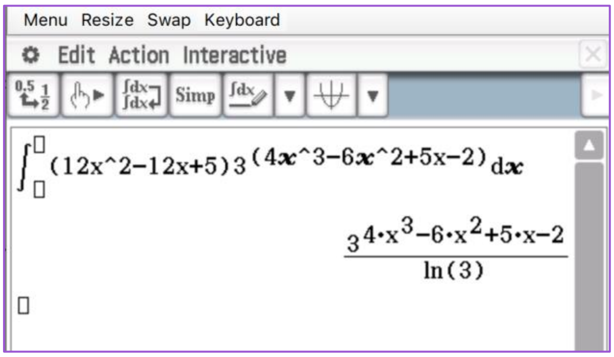
$$f'(x) = \frac{d}{dx}f(x) = 12x^2 - 12x + 5$$

And hence: $\int f'(x)3^{f(x)}$

$$= \int (12x^2 - 12x + 5) \times 3^{(4x^3-6x^2+5x-2)}dx$$

$$= \frac{3^{(4x^3-6x^2+5x-2)}}{\ln(3)}$$

Now testing this on a calculator:



Screenshot from ClassPAD Calculator, giving the same answer as the formula (without the constant). This strengthens the validity of our conjecture.

SUMMARY

If given:

- a is some real, positive constant
- x is a variable that can take real values
- $\ln(x) = \log_e x =$ the natural logarithm of x
- $f(x)$ is some function of x
- $f'(x)$ is the derivative of $f(x)$
- c is some suitable constant (constant of integration)
- b is a real constant

Then:

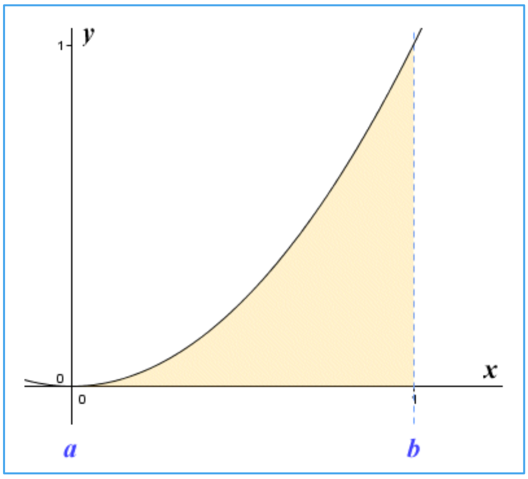
$$1.)\ \int a^x dx = \frac{a^x}{\ln(a)} + c$$

$$2.)\ \int a^{mx+b}\,dx = \frac{a^{mx+b}}{\ln(a^m)} + c$$

$$3.)\ \int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{\ln(a)} + c$$

And at any point in time if we require the area bound under the respective curves shown above from $p\ to\ q$, we may simply add in the **definite** integrals, with $p\ and\ q$ being the lower and upper bounds respectively, and continue the calculation as per usual.

CONCLUSION



In conclusion, this report has featured the integral calculus of exponential functions in which the base is not e. This research topic has linked strongly with the stimulus provided of "To infinity and beyond" as both ideas are strongly intertwined with the concept of infinity, as exponential functions are the most famous out of all functions for their ability to rise (or decline) to infinity so rapidly, and integration is based on the sum of infinitesimally small increments in a function. When we find the integral of an exponential function rising "to infinity", we sum its small increments also "to infinity". When brought together, both concepts easily cover the "and beyond" part as well as infinity plus infinity can only be beyond infinity!

Throughout this investigation, I have learnt a lot about both concepts studied; exponential functions and integration. I have also learnt how to better write reports, and have learned how to research better.

Some recommendations I would make for the future, for the betterment of this report is to perhaps include a wider range of research questions, which cover much more complex scenarios of this concept. I could also provide more numerical examples in order to prove the conjectures by further strengthening them.

However, I hope that I have done justice to the wonderful opportunity provided to me, and that the reader has learned something new.

I hope that this report has been successful in:

- Introducing exponential functions, and the process of integration, along with relevant formulas and laws required for the report.
- Providing complete, comprehensive, and clear answers to the above research questions, including full derivations and numerical examples where needed.
- Including the use of technology as screenshots (using Casio ClassPAD fx-CP400 graphic calculator)
- Providing full in-text referencing, along with professional mathematical format for working out.
- Including a comprehensive summary of results found.

Thank you for your time.

Tawqeer Saiyed.

REFERENCES

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